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Expanding and forwarding parameters of product graphs ☆

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Abstract

Expanding and forwarding are two graphic parameters related to the connectivity and the capacity of the network—the undirected graph with a given routing. Many large networks are composed from some existing smaller networks by using, in terms of graph theory, Cartesian product. The expanding and forwarding parameters of such large networks are associated strongly with that of the corresponding smaller ones. This association also provides a convenient way to determine the two parameters for some known networks such as the hypercube, generalized cube and the mesh, etc. As the generalization of the forwarding index, t-forwarding index is introduced and studied. The study shows that the t-forwarding parameters of a given graph are convergent (refers to the limit $t \to \infty$), which reveals some further properties concerning the forwarding parameters of the product graphs.

MSC: 05C90

Keywords: Routing; Expanding factor; Forwarding index; Product graph

1. Introduction and definitions

A network we consider here is defined to be an undirected graph with a given routing in advance, in which the vertices represent the nodes while the edges represent the links. In practice, the nodes are usually interpreted as computer/communication devices. Stimulated by the design of communication networks and distributed computer systems, a lot of graphic parameters were introduced and studied (see [8]). Expanding

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factor and forwarding index are two such parameters, which play important roles in the study of communication networks. Intuitively, expanding factors are measures of connectivity and the forwarding indices, the statistics on paths. These two parameters have been studied by many researchers (by algebraic, probabilistic and combinatorial techniques, for example, see [3,5–7,11,12]).

The reader is assumed to be familiar with the basic terminologies of graphs such as the adjacency, path, degree, induced subgraph and the spectrum of Laplasian of a graph, etc., for some other related definitions, we refer to [1,2]. All graphs discussed here are simple, undirected and connected. For a graph G, we denote by $E(G)$ and $V(G)$ the edge-set and vertex-set, respectively. And denote by $P=x_1x_2\cdots x_n$ the path from x_1 to x_n passing through $x_1,x_2,\ldots,x_n$. An edge connecting x and y will be denoted by $[xy]$. Let G be a graph and let $X\in V(G)$, $\bar{X}=V(G)-X$, the *edge-cut* of G induced by X is defined by

$$\partial X=\{[uv]\colon\ [uv]\in E(G), u\in X; v\in\bar{X}\}.$$

The *edge expanding factor* [10] is defined to be

$$\beta(G)=\min\left\{\frac{|\partial X|}{|X||\bar{X}|}\colon\ X\subset V(G), 1\leqslant|X|\leqslant|V(G)|-1\right\}.$$

An edge-cut where this minimum is met with equality is called *optimal*.

Let X be a proper subset of the vertex-set $V(G)$ of a graph G. The *vertex-cut* induced by X is

$$N(X)=\{v\in V(G)-X\colon\ [uv]\in E(G), u\in X\}.$$

The *vertex expanding factor* [10] of a graph G is defined similarly by

$$\gamma(G)=\min\left\{\frac{|N(X)|}{|X||X^+|}\colon\ X\subset V(G), 1\leqslant|X|\leqslant|V(G)|-1, |X^+|\geqslant 1\}\right\},$$

where X^+ denotes the complement of $X\cup N(X)$ in $V(G)$.

A *routing* R of a graph G is a set of paths connecting each ordered pair of distinct vertices of $V(G)$. The load $R([uv])$ of an edge $[uv]\in E(G)$ in R is the number of paths of R passing through the edge $[uv]$. The edge forwarding index of G, introduced firstly by Chung et al. [3], is

$$\pi(G)=\min_{R}\max_{[uv]\in E(G)}R([uv]).$$

A routing where this minimum is met with equality is called *edge optimal*. Similarly, the *load* $R(u)$ of a vertex $u\in V(G)$ is the number of paths of R admitting u as inner vertex. The *vertex forwarding index* $\xi(G)$ of G is defined by

$$\xi(G)=\min_{R}\max_{u\in V(G)}R(u).$$

A routing that meets this minimum is called *vertex optimal*.

Let R be a routing, and let u, v be two vertices. Denote by $R(u,v)$ the path of R connecting u and v (ordered pair).

In [11], Sole firstly established a connection between expanding factors and forwarding indices (vertex and edge) by showing a simple inequality. So far these two parameters were determined only for a few graph classes such the paths, trees, cycles and hypercubes, etc. [3,4,6,7,11,12]. For some other graphs (for example, Johnson graph, de bruijn graph and Moore graph, etc.), several partial results were also obtained. On the other hand, these two parameters are still unknown for many popular network topologies.

In practice it is desirable to be able to compose larger networks by using one or more existing networks as building blocks. A natural and frequently used way is by using the *Cartisian product* (or called Cartisian sum, see [6]): $G_1 \times G_2$, where G_1 and G_2 are two given graphs and

$$V(G_1 \times G_2) = \{\{u,v\}\colon\ u \in V(G_1), v \in V(G_2)\}$$

and

$$\begin{aligned} E(G_1 \times G_2) = \{[\{u,v\}\{u',v'\}]\colon\ & u = u' \text{ and } [vv'] \in E(G_2); \\ & \text{or } v = v' \text{ and } [uu'] \in E(G_1)\}. \end{aligned}$$

For examples, the hypercube, k-cube $K_{d,n}$ (or called generalized cube) and the mesh (see [9]), etc., are constructed in such a way.

In this paper, our main aim is to study the expanding and forwarding parameters of the graphs produced by Cartisian product (or product graph, for short). To determine the forwarding and expanding parameters of a product graph $G_1 \times G_2$, a natural way is to find the connection for these parameters, between $G_1 \times G_2$ and G_1, G_2. More precisely, is $\beta(G_1 \times G_2)$ (also the other parameters) determined uniquely by $\beta(G_1), \beta(G_2)$ and the orders of G_1 and G_2? Heydemann et al. [6] firstly did some work on this problem. They showed that

$$\pi(G_1 \times G_2) \leqslant \max\{|V(G_1)|\pi(G_2), |V(G_2)|\pi(G_1)\}$$

and

$$\xi(G_1 \times G_2) \leqslant |V(G_2)|\xi(G_1) + |V(G_1)|\xi(G_2) + (|V(G_1)| - 1)(|V(G_2)| - 1).$$

For $\beta(G_1 \times G_2)$, in the following section we will show that the answer to the above question is Yes. For other parameters, we also give partial answers by showing several upper and lower bounds (by using combinatorial and algebraic techniques).

Let $t \geqslant 1$ be a natural number, a t-routing R^t is defined to be the set of paths connecting each ordered pair of vertices by exactly t paths. The t-forwarding indices

$\pi^t(G)$ and $\xi^t(G)$ of a graph G are defined similarly by

$$\pi^t(G) = \frac{1}{t}\min_{R^t}\max_{[uv]\in E(G)} R^t([uv]) \quad \text{and} \quad \xi^t(G) = \frac{1}{t}\min_{R^t}\max_{u\in V(G)} R^t(u),$$

where the loading $R^t([uv])$ and $R^t(u)$ of an edge $[uv]$ and a vertex u are defined analogously as $R([uv])$ and $R(u)$. As one see, the t-forwarding index is a natural generalization of the common forwarding index and hence, π and ξ are π^1 and ξ^1, respectively. In the Section 3, some basic results concerning the t-forwarding indices are obtained, one of which improves the previous result of Heydemann et al. [6] about the forwarding index of product graph.

The last section is an application of the former sections, we will determine the expanding and forwarding parameters for some popular networks.

2. Expanding factors

For convenience, In what follows we will use $n_1, n_2, \beta_1, \beta_2, \pi_1, \pi_2, \gamma_1, \gamma_2$ and ξ_1, ξ_2 to represent $|V(G_1)|, |V(G_2)|, \beta(G_1), \beta(G_2), \pi(G_1), \pi(G_2), \gamma(G_1), \gamma(G_2)$ and $\xi(G_1), \xi(G_2)$, respectively.

Theorem 2.1. *For any two graphs G_1 and G_2,*

$$\beta(G_1 \times G_2) = \min\left\{\frac{\beta_2}{n_1}, \frac{\beta_1}{n_2}\right\}.$$

To prove the theorem, we need some necessary notations and Lemma. Let m, n, and s, $0 \leqslant s \leqslant m \times n$, be three positive integers. An (m,n,s)-chessboard S is defined to be a $m \times n$-chessboard with s unit squares being filled with $*$'s. As an example, a $(10,15,60)$-chessboard is illustrated in Table 1.

Denote by $\mathscr{S}(m,n,s)$ the class of all (m,n,s)-chessboards. Let $S \in \mathscr{S}(m,n,s)$, denote by $x_i(S)$ the number of $*$'s in the ith row of S, $i = 1,2,\ldots,m$; and denote by $y_j(S)$ the number of $*$'s in the jth column of S, $j = 1,2,\ldots,n$. Usually, $x_i(S)$ and $y_j(S)$ are

Table 1
An $(10,15,60)$-chessboard S

x_{10}			*												
.											*				
.	*			*											
.	*	*	*												
.			*	*					*	*					*
.	*	*	*			*	*	*				*			
.			*		*		*	*	*		*	*			
.	*		*			*	*			*	*	*	*		*
x_2	*	*	*	*		*	*	*	*			*	*		*
x_1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	
	y_1	y_2	.	.	.	.	.	.	.	.	.	.	.	.	y_{15}

Table 2
The corresponding chessboard S' of S

x_{10}	*														
.	*														
.	*	*													
.	*	*	*												
.	*	*	*	*	*										
.	*	*	*	*	*	*	*								
.	*	*	*	*	*	*	*								
.	*	*	*	*	*	*	*	*	*						
x_2	*	*	*	*	*	*	*	*	*	*	*				
x_1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	
	y_1	y_2	.	.	.	.	.	.	.	.	.	.	.	.	y_{15}

simply written by x_i, y_j, with no confusions. Let β_1, β_2 be two positive (real) numbers, define

$$\rho(S,\beta_1,\beta_2) = \beta_1 \sum_{i=1}^{m} x_i^2 + \beta_2 \sum_{j=1}^{n} y_j^2.$$

Lemma 2.1. *For any* $S \in \mathscr{S}(m,n,s)$,

$$\rho(S,\beta_1,\beta_2) \leqslant \max\left\{\beta_2 ms + \frac{\beta_1 s^2}{m}, \frac{\beta_2 s^2}{n} + \beta_1 ns\right\}.$$

Proof. Because when we rearrange the columns and the rows of S, the resulting chessboard does not change the value of $\rho(S,\beta_1,\beta_2)$, we may make an assumption: $n \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m \geqslant 0$ and $m \geqslant y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n \geqslant 0$. Let S' be the (m,n,s)-chessboard obtained from S by moving the $*$'s in the ith row of $\mathscr{S}$ to the first x_i unit square of the ith row. For an example, the resulting chessboard S' of S in Table 1 is illustrated as in Table 2.

It is easy to verify that $\rho(S,\beta_1,\beta_2) \leqslant \rho(S',\beta_1,\beta_2)$ and furthermore, $m \geqslant y_1(S') \geqslant y_2(S') \geqslant \cdots \geqslant y_n(S') \geqslant 0$. Because of the symmetry, we can assume, without loss of generality, that $m\beta_2 \geqslant n\beta_1$. We now prove

$$\rho(S',\beta_1,\beta_2) \leqslant \beta_2 ms + \frac{\beta_1 s^2}{m},$$

by induction on m and n.

When $m = 1$ or $n = 1$, the assertion holds immediately. Let the induction go step further.

Case 1: $s \leqslant n$

Subcase 1.1: $y_n(S') = 0$. In this case, $\rho(S',\beta_1,\beta_2) = \beta_1 \sum_{i=1}^{m} x_i^2 + \beta_2 \sum_{j=1}^{n} y_j^2 = \beta_1 \sum_{i=1}^{m} x_i^2 + \beta_2 \sum_{j=1}^{n-1} y_j^2$. That is, S' can be considered as a $(m,n-1,s)$-chessboard. On the other hand, note that $m\beta_2 \geqslant n\beta_1$, then $m\beta_2 \geqslant (n-1)\beta_1$. So by inductive hypothesis

$$\rho(S', \beta_1, \beta_2) \leqslant \beta_2 ms + \frac{\beta_1 s^2}{m}.$$

Subcase 1.2: $y_n(S') = 1$. In this case, $y_1 = y_2 = \cdots = y_n = 1$ and $x_2 = x_3 = \cdots = x_m = 0$, $x_1 = n = s$. Since $m\beta_2 \geqslant n\beta_1$ and $m \geqslant 1$, then by a direct enumeration,

$$\rho(S', \beta_1, \beta_2) = \beta_1 n^2 + n\beta_2 \leqslant \beta_2 ms + \frac{\beta_1 s^2}{m}.$$

Case 2: $s > n$.

Subcase 2.1: $y_n = 0$. This case is similar to Case 1.1.

Subcase 2.2: $y_n > 0$. In this case $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n \geqslant 1$. Denote by S'_{m-1} the $(m-1, n, s-n)$-chessboard obtained from S' by deleting the last (the mth) row. Let x'_i be the number of $*$'s in the ith row of S'_{m-1}, $i = 1, 2, \ldots, m-1$, and let y'_j be the number of $*$'s in the jth column of S'_{m-1}, $j = 1, 2, \ldots, n$. Then $x'_i = x_i$, $i = 1, 2, \ldots, m-1$, $x_m = n$ and $y'_j = y_j - 1$, $j = 1, 2, \ldots, n$, $\sum_{i=1}^{m-1} x'_i = \sum_{j=1}^{n} y'_j = s - n$. Hence

$$\rho(S', \beta_1, \beta_2) = \beta_1 \sum_{i=1}^{m} x_i^2 + \beta_2 \sum_{j=1}^{n} y_j^2 = \beta_1 \sum_{i=1}^{m-1} (x'_i)^2 + \beta_2 \sum_{j=1}^{n} (y'_j)^2 + \beta_1 n^2 + 2\beta_2(s-n) + n\beta_2. \tag{1}$$

Subcase 2.2.1: $(m-1)\beta_2 \geqslant n\beta_1$. By the inductive hypothesis

$$\rho(S'_{m-1}, \beta_1, \beta_2) = \beta_1 \sum_{i=1}^{m-1} (x'_i)^2 + \beta_2 \sum_{j=1}^{n} (y'_j)^2 \leqslant \beta_2(m-1)(s-n) + \frac{\beta_1 (s-n)^2}{m-1}. \tag{2}$$

Combine (2) with (1), we have

$$\begin{aligned}\rho(S', \beta_1, \beta_2) &\leqslant \beta_2(m-1)(s-n) + \beta_1 \frac{(s-n)^2}{m-1} + \beta_1 n^2 + 2\beta_2(s-n) + n\beta_2 \\ &= \beta_2 ms + \beta_1 \frac{(s-n)^2}{m-1} + \beta_2 s + \beta_1 n^2 - \beta_2 mn.\end{aligned}$$

Let

$$\begin{aligned}\phi(s) &= \beta_2 ms + \beta_1 \frac{s^2}{m} - \rho(S', \beta_1, \beta_2) \\ &\geqslant \frac{1}{m(m-1)} (2\beta_1 mns + m\beta_2 s + n\beta_2 m^3 + n^2\beta_1 m - \beta_1 s^2 - \beta_1 mn^2 \\ &\quad - m^2\beta_2 s - n\beta_2 m^2 - n^2\beta_1 m^2).\end{aligned}$$

Then, $\phi(s)$ is a quadratic function in s which reaches the minimum value when $s = mn$ $(n \leqslant s \leqslant mn)$. That is,

$$\phi(s) \geqslant \min\{\phi(n), \phi(mn)\} = \phi(mn) = 0.$$

Subcase 2.2.2: $(m-1)\beta_2 \leqslant n\beta_1$. Change the roles of m and n from each other, and by the inductive hypothesis,

$$\rho(S'_{m-1},\beta_1,\beta_2) \leqslant \beta_2\frac{(s-n)^2}{n} + \beta_1 n(s-n).$$

Hence

$$\begin{aligned}\rho(S',\beta_1,\beta_2) &\leqslant \beta_2\frac{(s-n)^2}{n} + \beta_1 n(s-n) + \beta_1 n^2 + 2\beta_2(s-n) + n\beta_2 \\ &= \beta_2\frac{s^2}{n} + \beta_1 ns \leqslant \beta_2 ms + \beta_1\frac{s^2}{m}.\end{aligned}$$

The last inequality holds because $m\beta_2 \geqslant n\beta_1$ and $mn \geqslant s$. The lemma now follows. □

Proof of Theorem 2.1. Let ∂X_1 and ∂X_2 be two optimal edge-cut of G_1 and G_2, respectively. Let

$$X_{12} = \{\{x_1,x_2\}\colon x_1 \in X_1, x_2 \in V(G_2)\}$$

and

$$X_{21} = \{\{x_1,x_2\}\colon x_1 \in V(G_1), x_2 \in X_2\}.$$

Then one can see that the edge-cuts ∂X_{12} and ∂X_{21} of $G_1 \times G_2$ satisfy:

$$\frac{|\partial X_{12}|}{|X_{12}||\bar{X}_{12}|} = \frac{|\partial X_1|n_2}{|X_1|n_2\cdot|\overline{X_1}|n_2} = \frac{\beta_1}{n_2} \quad\text{and}\quad \frac{|\partial X_{21}|}{|X_{21}||\bar{X}_{21}|} = \frac{|\partial X_2|n_1}{|X_2|n_1\cdot|\overline{X_2}|n_1} = \frac{\beta_2}{n_1},$$

which implies that $\beta(G_1 \times G_2) \leqslant \min\{\beta_2/n_1, \beta_1/n_2\}$.

Now we show $\beta(G_1 \times G_2) \geqslant \min\{\beta_2/n_1, \beta_1/n_2\}$. For convenience, let $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Let ∂X be an edge-cut of $G_1 \times G_2$ (induced by X). For any $i \in \{1,2,\ldots,n_2\}$, $j \in \{1,2,\ldots,n_1\}$, denote

$$X_{1i} = \{\{u,v_i\}\colon \{u,v_i\} \in X\}; \quad X_{2j} = \{\{u_j,v\}\colon \{u_j,v\} \in X\}$$

and

$$E_{1i} = \{[\{u,v_i\}\{u',v_i\}] \in \partial X\colon [uu'] \in E(G_1)\};$$

$$E_{2j} = \{[\{u_j,v\}\{u_j,v'\}] \in \partial X\colon [vv'] \in E(G_2)\}.$$

Then it is easy to check that

$$X = \bigcup_{i=1}^{n_2} X_{1i} = \bigcup_{j=1}^{n_1} X_{2j} \quad\text{and}\quad \partial X = \left(\bigcup_{i=1}^{n_2} E_{1i}\right) \cup \left(\bigcup_{j=1}^{n_1} E_{2j}\right).$$

Let $x_i = |X_{1i}|, y_j = |X_{2j}|$ and $e_{1i} = |E_{1i}|$, $e_{2j} = |E_{2j}|$, $i \in \{1,2,\ldots,n_2\}$, $j \in \{1, 2,\ldots,n_1\}$. Since β_1 and β_2 are the edge expanding factors of G_1 and G_2, respectively.

Then

$$\frac{|E_{1i}|}{|X_{1i}||X_i(G_1)-X_{1i}|}=\frac{e_{1i}}{x_i(n_1-x_i)}\geqslant\beta_1 \quad \text{and}$$

$$\frac{|E_{2j}|}{|X_{2j}||X_j(G_2)-X_{2j}|}=\frac{e_{2j}}{y_j(n_2-y_j)}\geqslant\beta_2,$$

where

$$X_i(G_1)=\{\{u,v_i\}\colon\ u\in V(G_1)\} \quad \text{and} \quad X_j(G_2)=\{\{u_j,v\}\colon\ v\in V(G_2)\}.$$

Hence,

$$\begin{aligned}\frac{|\partial X|}{|X||\bar{X}|}&=\frac{\sum_{i=1}^{n_2}e_{1i}+\sum_{j=1}^{n_1}e_{2j}}{(\sum_{i=1}^{n_2}x_i)(n_1n_2-\sum_{i=1}^{n_2}x_i)}\\&\geqslant\frac{\beta_1\sum_{i=1}^{n_2}x_i(n_1-x_i)+\beta_2\sum_{j=1}^{n_1}y_j(n_2-y_j)}{(\sum_{i=1}^{n_2}x_i)(n_1n_2-\sum_{i=1}^{n_2}x_i)}\\&=\frac{n_1\beta_1\sum_{i=1}^{n_2}x_i+n_2\beta_2\sum_{j=1}^{n_1}y_j-(\beta_1\sum_{i=1}^{n_2}x_i^2+\beta_2\sum_{j=1}^{n_1}y_j^2)}{(\sum_{i=1}^{n_2}x_i)(n_1n_2-\sum_{i=1}^{n_2}x_i)}.\end{aligned}$$

For any $\{u_i,v_j\}\in V(G_1\times G_2)$, we may regard it as the unit square which lies in the ith column and the jth column of the $n_2\times n_1$-chessboard. Furthermore, if $\{u_i,v_j\}\in X$, then we fill a $*$ into the corresponding square. Thus each edge-cut ∂X corresponds to a (n_2,n_1,s)-chessboard, where $s=|X|=\sum_{i=1}^{n_2}x_i=\sum_{j=1}^{n_1}y_j$.

Without loss of generality, we may assume $\beta_1/n_2\geqslant\beta_2/n_1$, i.e. $n_1\beta_1\geqslant n_2\beta_2$. So by Lemma 2.1,

$$\frac{|\partial X|}{|X||\bar{X}|}\geqslant\frac{n_2\beta_2-\frac{1}{n_1}\beta_2\sum_{i=1}^{n_2}x_i}{(n_1n_2-\sum_{i=1}^{n_2}x_i)}=\frac{\beta_2}{n_1}$$

which completes the proof of the theorem. □

Given two natural numbers $n>k\geqslant 1$, we define

$$\beta_{k,n}=\max_{|V(G)|=n,\Delta(G)=k}\beta(G),$$

where $\Delta(G)$ is the maximum degree of G. A graph where this maximum is met with equality is called expanding optimal. It is not difficult to verify that $\beta_{2,4}=1$ with expanding optimal graph C_4: the cycle of order 4. The following result is a direct corollary of Theorem 2.1.

Corollary 2.1. *Let $n>k\geqslant 1$, then*

$$\beta_{k,n}\geqslant\max_{n_1n_2=n,k_1+k_2=k}\min\left\{\frac{\beta_{k_1,n_1}}{n_2},\frac{\beta_{k_2,n_2}}{n_1}\right\},$$

where $n_1>k_1\geqslant 1$ and $n_2>k_2\geqslant 1$.

For vertex expanding factor, we have the following theorem.

Theorem 2.2. *Let G_1 and G_2 be two graphs, then*

$$\gamma(G_1 \times G_2) \leqslant \min\left\{\frac{\gamma_1}{n_2}, \frac{\gamma_2}{n_1}\right\}.$$

Proof. Let $N(X_1)$ and $N(X_2)$ be two optimal vertex-cuts (induced by X_1 and X_2) of G_1 and G_2, respectively. As in the proof of Theorem 2.1, let

$$X_{12} = \{\{x_1, x_2\}: x_1 \in X_1, x_2 \in V(G_2)\}$$

and

$$X_{21} = \{\{x_1, x_2\}: x_1 \in V(G_1), x_2 \in X_2\}.$$

Then the vertex-cut $N(X_{12})$ and $N(X_{21})$ of $G_1 \times G_2$ satisfy:

$$\frac{|N(X_{12})|}{|X_{12}||X_{12}^+|} = \frac{|N(X_1)|n_2}{|X_1|n_2 \cdot |X_1^+|n_2} = \frac{\gamma_1}{n_2} \quad \text{and} \quad \frac{|N(X_{21})|}{|X_{21}||X_{21}^+|} = \frac{|N(X_2)|n_1}{|X_2|n_1 \cdot |X_2^+|n_1} = \frac{\gamma_2}{n_1}.$$

The theorem now follows. $\square$

3. Forwarding indices

3.1. Upper bounds and lower bounds

We start this section with the following two results obtained by Sole in [11], one of which establishes a connection between the forwarding and the expanding parameters, the other gives a lower bound for the edge forwarding index of a graph G, in terms of its smallest nonzero eigenvalue of the Laplasian and the maximum degree:

Lemma 3.1.1 (Sole [11]). *For any connected graph G,*

$$\pi(G)\beta(G) \geqslant 2 \quad \textit{and} \quad \gamma(G)\xi(G) \geqslant 2.$$

Lemma 3.1.2 (Sole [11]). *If the smallest nonzero eigenvalue of Laplasian of a graph G of order n is λ and the maximum degree of G is Δ, then*

$$\pi(G) \geqslant \frac{n}{\sqrt{n(2\Delta - \lambda)}} \quad \textit{and} \quad \gamma(G) \geqslant \frac{4\lambda}{n\Delta}.$$

Contrast to the edge expanding factor, it seems difficult to represent $\xi(G_1 \times G_2)$, $\pi(G_1 \times G_2)$ by (if possible) $\xi_1, \xi_2; \pi_1, \pi_2$ and n_1, n_2, respectively. In [6], Heydemann et al. established the following two upper bounds:

Theorem 3.1.1 (Heydemann et al. [6]).

$$\pi(G_1 \times G_2) \leqslant \max\{\pi_1 n_2, \pi_2 n_1\} \tag{3}$$

and

$$\xi(G_1 \times G_2) \leqslant \xi_2 n_1 + (n_1 - 1)(n_2 - 1) + \xi_1 n_2.$$

By Theorems 2.1, 3.1.1 and Lemma 3.1.1, we can also give a sufficient condition for equality in (3) (the proof is obvious, we omit it from the paper).

Corollary 3.1.1. *If* $\beta_i \pi_i = 2,\ i = 1,2$, *then*

$$\pi(G_1 \times G_2) = \max\{\pi_1 n_2, \pi_2 n_1\}.$$

The following lower bounds are obtained directly from Theorems 2.1, 2.2 and Lemma 3.1.1.

Corollary 3.1.2.

$$\pi(G_1 \times G_2) \geqslant \max\left\{\frac{2n_1}{\beta_2}, \frac{2n_2}{\beta_1}\right\} \quad \textit{and} \quad \xi(G_1 \times G_2) \geqslant \min\left\{\frac{2n_1}{\gamma_2}, \frac{2n_2}{\gamma_1}\right\}.$$

Using algebraic techniques (spectrum of the Laplacian), we can also obtain two other lower bounds.

Corollary 3.1.3. *If the smallest nonzero eigenvalue of Laplasian of* G_i *is* λ_i *and the maximum degree of* G_i *is* Δ_i, $i = 1,2$, *then*

$$\pi(G_1 \times G_2) \geqslant \frac{n_1 n_2}{\sqrt{n_1 n_2 (2(\Delta_1 + \Delta_2) - \min\{\lambda_1, \lambda_2\})}}.$$

Proof. By a theorem in [5], we know that the Laplasian eigenvalues of the product graph $G_1 \times G_2$ are equal to all the possible sums of the eigenvalues of G_1 and G_2. On the other hand, we know that any graph has 0 as its eigenvalue while all other eigenvalues are positive (see [2]). Thus, the smallest nonzero eigenvalue of $G_1 \times G_2$ is $\min\{\lambda_1, \lambda_2\}$. Noting that the maximal degree of $G_1 \times G_2$ is $\Delta_1 + \Delta_2$, the theorem follows from Lemma 3.1.2. □

The following corollary is obtained by using the same idea as above.

Corollary 3.1.4. *Let* G_1 *and* G_2 *be as in Corollary* 3.1.3, *then*

$$\gamma(G_1 \times G_2) \geqslant \frac{4\min\{\lambda_1, \lambda_2\}}{n_1 n_2 (\Delta_1 + \Delta_2)}.$$

For any two given natural numbers $n > k \geqslant 1$, define

$$\pi_{k,n} = \min_{|V(G)|=n, \Delta(G)=k} \pi(G) \quad \text{and} \quad \xi_{k,n} = \min_{|V(G)|=n, \Delta(G)=k} \xi(G).$$

In [3], Chung et al. introduced the forwarding index problem: find a graph G of order n and maximum degree k such that $\xi(G)=\xi_{k,n}$. By Theorem 3.1.1, we have the following result.

Corollary 3.1.5. *Let $n > k \geqslant 1$, then*

$$\pi_{k,n} \leqslant \min_{n_1 n_2=n, k_1+k_2=k} \max\{n_2\pi_{k_1,n_1}, n_1\pi_{k_2,n_2}\}$$

and

$$\xi_{k,n} \leqslant \min_{n_1 n_2=n, k_1+k_2=k} \max\{n_2\xi_{k_1,n_1}, n_1\xi_{k_2,n_2}\},$$

where $n_1 > k_1 \geqslant 1$ and $n_2 > k_2 \geqslant 1$.

3.2. t-forwarding indices

By a direct observation on the definition of t-forwarding index, we see that for any natural number $k \geqslant 1$, each t-routing R^t will induce a kt-routing R^{kt} by replacing each path P in R^t by k copies of P. In other words, for any graph G and natural number $t \geqslant 1$,

$$\pi(G) \geqslant \pi^t(G) \geqslant \pi^{t^2}(G) \geqslant \cdots \geqslant \pi^{t^k}(G)\cdots \tag{4}$$

and

$$\xi(G) \geqslant \xi^t(G) \geqslant \xi^{t^2}(G) \geqslant \cdots \geqslant \xi^{t^k}(G)\cdots \ . \tag{5}$$

Note that for any connected graph G and the natural number $m \geqslant 1$, $\pi^m(G), \xi^m(G) > 0$. So $\pi^{t^k}(G)$ and $\xi^{t^k}(G)$ are convergent (refers to the limit $k \to \infty$ with t fixed) to two constants, say $\pi^0(G,t)$ and $\xi^0(G,t)$, respectively. In general, we have the following stronger result.

Theorem 3.2.1. *For any graph G,*

$$\lim_{t\to\infty} \pi^t(G) = \pi^0(G) \quad \text{and} \quad \lim_{t\to\infty} \xi^t(G) = \xi^0(G),$$

where $\pi^0(G)$ and $\xi^0(G)$ are two constants determined uniquely by G.

Proof. We only give the proof for the vertex forwarding index (the proof for the edge forwarding index is similar). For simplicity, we rewrite $\pi^t(G)$ and $\xi^t(G)$ as π^t and ξ^t, respectively.

It is sufficient to prove that for any $\varepsilon > 0$, there is a natural number $N(\varepsilon)$ such that for any $n \geqslant N(\varepsilon)$, $|\xi^n - \xi^0| < \varepsilon$.

Let n_1, n_2 and $n = n_1 + n_2$ be the natural numbers and let R^n, R^{n_1} and R^{n_2} be the optimal n-routing, n_1-routing and n_2-routing of G, respectively. Note that $R^{n_1} \cup R^{n_2}$ is also a n-routing of G, so

$$\max_{u\in G}(R^{n_1}(u) + R^{n_2}(u)) \geqslant \max_{u\in G} R^n(u).$$

This implies that

$$\xi^n \leqslant \frac{n_1}{n}\xi^{n_1} + \frac{n_2}{n}\xi^{n_2}. \tag{6}$$

Recall that $\xi^{2^k} \to \xi^0(G,2)$ (as $k \to \infty$). We choose $N(\varepsilon) = 2^{N+1}$, where N satisfies

$$N \geqslant \max\{N_1, N_2\colon\ 2^{-N_1/2}|\xi - \xi^0| < \varepsilon/4,\ |\xi^{2^{N_2}} - \xi^0| < \varepsilon/4\},$$

where $\xi^0 = \xi^0(G,2)$. Let $n \geqslant N(\varepsilon)$. For convenience, we write

$$n = c_0 + c_1 2 + c_2 2^2 + \cdots + c_k 2^k,$$

where $c_i \in \{0,1\}$, $i = 0,1,2,\ldots,k-1$, and $c_k = 1$. Since $n \geqslant N(\varepsilon) = 2^{N+1}$, then $k \geqslant N$. And noting that $\xi^n \geqslant \xi^0$, so by (6) and (5)

$$\begin{aligned}
0 \leqslant \xi^n - \xi^0 &= \frac{1}{n}(c_0\xi + c_1 2\xi^2 + \cdots + c_k 2^k \xi^{2^k} - n\xi^0)\\
&= \frac{1}{n}(c_0(\xi - \xi^0) + c_1 2(\xi^2 - \xi^0) + \cdots + c_k 2^k(\xi^{2^k} - \xi^0))\\
&\leqslant \frac{1}{n}((\xi - \xi^0) + 2(\xi^2 - \xi^0) + \cdots + 2^k(\xi^{2^k} - \xi^0))\\
&\leqslant \frac{1}{n}((\xi - \xi^0) + 2(\xi - \xi^0) + \cdots + 2^{N/2-1}(\xi - \xi^0)\\
&\quad + 2^{N/2}(\xi^{2^{N/2}} - \xi^0) + \cdots + 2^k(\xi^{2^{N/2}} - \xi^0))\\
&= \frac{1}{n}((2^{N/2} - 1)(\xi - \xi^0) + (2^{k+1} - 2^{N/2})(\xi^{2^{N/2}} - \xi^0)) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The theorem now follows. □

Using the same idea as the proof of Lemma 3.1.1 (see Theorems 1 and 7 in [11]) and combining with (4) and Theorem 3.2.1, we have

Corollary 3.2.1. *For any connected graph G and natural number $t \geqslant 1$,*

$$\beta(G)\pi(G) \geqslant \beta(G)\pi^t(G) \geqslant \beta(G)\pi^0(G) \geqslant 2.$$

Compare with Theorem 3.1.1, a further result is shown by the following.

Theorem 3.2.3. *For any two graphs G_1 of order m and G_2 of order n,*

$$\begin{aligned}
\max\{n\pi(G_1), m\pi(G_2)\} &\geqslant \max\{n\pi^n(G_1), m\pi^m(G_2)\}\\
&\geqslant \pi(G_1 \times G_2) \geqslant \max\{n\pi^{n^2}(G_1), m\pi^{m^2}(G_2)\}
\end{aligned} \tag{7}$$

and

$$n\xi(G_1)+(m-1)(n-1)+m\xi(G_2) \geqslant n\xi^n(G_1)+(m-1)(n-1)+m\xi^m(G_2)$$
$$\geqslant \xi(G_1\times G_2). \qquad (8)$$

Proof. We first give the proof for (8). Let $V(G_1)=\{u_1,u_2,\dots,u_m\}$ and $V(G_2)=\{v_1,v_2,\dots,v_n\}$. Let R^n and R^m be two optimal n- and m-routing of G_1 and G_2, respectively. For any $i,j\in\{1,2,\dots,m\},\ i\neq j$, denote by $P_1(i,j),P_2(i,j),\dots,P_n(i,j)$ the paths of R^n connecting from the vertex u_i to u_j. Similarly, denote by $S_1(i,j),S_2(i,j),\dots,S_m(i,j)$ the paths of R^m connecting from the vertex v_i to v_j.

We notice that $G_1\times G_2$ consists of n copies of G_1 (also m copies of G_2) induced by the vertex-subset: $\{\{u_1,v_k\},\{u_2,v_k\},\dots,\{u_m,v_k\}\},\ k=1,2,\dots,n$, respectively. For convenience, let us denote them by $G_1^k,\ k=1,2,\dots,n$. Denote by $P_l^k(i,j)$ the corresponding path of $P_l(i,j)$ in the copy G_1^k of $G_1,\ l,k\in\{1,2,\dots,n\}$. The symbols G_2^i and $S_l^k(i,j)$ are defined analogously.

Define R to be the routing of $G_1\times G_2$ for two vertices $\{u_i,v_j\},\{u_s,v_t\}\in V(G_1\times G_2)$,

Case 1: $j=t$. Then the path $R(\{u_i,v_j\},\{u_s,v_t\})=P_j^j(i,s)$.

Case 2: $i=s$. Then $R(\{u_i,v_j\},\{u_s,v_t\})=S_i^i(j,t)$.

Case 3: $i\neq s$ and $j\neq t$. Then $R(\{u_i,v_j\},\{u_s,v_t\})=P_t^j(i,s)\cup S_i^s(j,t)$ (noting that $P_t^j(i,s)$ and $S_i^s(j,t)$ have exactly one vertex $\{s,j\}$ in common, so $P_t^j(i,s)\cup S_i^s(j,t)$ is a path in $G_1\times G_2$ which connects $\{u_i,v_j\}$ and $\{u_s,v_t\}$).

Thus, for any vertex $\{u_i,v_j\}\in V(G_1\times G_2)$, the paths P admitting $\{u_i,v_j\}$ as an inner vertex can be partitioned into the following three types.

Type 1: $P=\{u_l,v_j\}\cdots\{u_i,v_j\}\cdots\{u_s,v_j\}\cdots\{u_s,v_t\}$, where the subpath $\{u_l,v_j\}\cdots\{u_i,v_j\}\cdots\{u_s,v_j\}\in G_1^j$ and $\{u_s,v_j\}\cdots\{u_s,v_t\}\in G_2^s$.

Type 2: $P=\{u_l,v_s\}\cdots\{u_i,v_s\}\cdots\{u_i,v_j\}\cdots\{u_i,v_t\}$, where the subpath $\{u_l,v_s\}\cdots\{u_i,v_s\}\in G_1^s$ and $\{u_i,v_s\}\cdots\{u_i,v_j\}\cdots\{u_i,v_t\}\in G_2^i$.

Type 3: $P=\{u_l,v_j\}\cdots\{u_i,v_j\}\cdots\{u_i,v_t\}$, where the subpath $\{u_l,v_j\}\cdots\{u_i,v_j\}\in G_1^j$ and $\{u_i,v_j\}\cdots\{u_i,v_t\}\in G_2^i$.

By the definition of R, one can check that the numbers of paths in Types 1–3 are $R^n(u_i)$, $R^m(v_j)$ and $(m-1)(n-1)$, respectively. In other words,

$$\xi(G_1\times G_2)\leqslant n\xi^n(G_1)+m\xi^m(G_2)+(m-1)(n-1).$$

The first "$\geqslant$" in (8) is from (5).

We now give the proof for (7). Let R^n, R^m and R be defined as above. And let $[\{u_i,v_j\}\{u_s,v_t\}]$ be an edge of $G_1\times G_2$, then $i=s$ or $j=t$.

Case 1: $j=t$. By the definition of R^n, R^m and R, we have

$$R([\{u_i,v_j\}\{u_s,v_t\}])=R^n([u_iu_s]).$$

Case 2: $i=s$. Similarly,

$$R([\{u_i,v_j\}\{u_s,n_t\}])=R^m([v_jv_t]).$$

From the above two cases we have

$$\begin{aligned}\pi(G_1\times G_2) &\leqslant \max_{[\{u_i,v_j\}\{u_s,v_t\}]\in E(G_1\times G_2)} R([\{u_i,v_j\}\{u_s,v_t\}])\\ &= \max\left\{\max_{[u_iu_s]\in E(G_1)} R^n([u_iu_s]),\ \max_{[v_jv_t]\in E(G_2)} R^m([v_jv_t])\right\}\\ &= \max\{n\pi^n(G_1), m\pi^m(G_2)\}.\end{aligned}$$

Now we prove the last "$\geqslant$" in (7) (the first "$\geqslant$" is immediate from (4)). Let P be a path of $G_1\times G_2$, we denote by $P_{G_1^i}$ and $P_{G_2^i}$ the projections of P on G_1^i and G_2^i, respectively. For instance, let $P=\{u_1,v_3\}\{u_2,v_3\}\{u_2,v_2\}\{u_2,v_4\}\{u_3,v_4\}\{u_5,v_4\}$, then $P_{G_1^i}=\{u_1,v_i\}\{u_2,v_i\}\{u_3,v_i\}\{u_5,v_i\}$ and $P_{G_2^i}=\{u_i,v_3\}\{u_i,v_2\}\{u_i,v_4\}$.

Let R be an optimal routing of $G_1\times G_2$, define

$$R^1=\{P_{G_1^1}\colon P\in R\}\quad\text{and}\quad R^2=\{P_{G_2^1}\colon P\in R\}.$$

It is easy to verify that R^1 and R^2 are an n^2-routing and an m^2-routing of the copies G_1^1 and G_2^1, respectively. Let $[\{u_i,v_j\}\{u_s,v_t\}]$ be an edge of $G_1\times G_2$. Then $i=s$ or $j=t$.

Case 1: $j=t$. By the definition of R^1 and R^2, we have

$$R([\{u_i,v_j\}\{u_s,v_t\}])\geqslant \frac{1}{n}R^1([\{u_i,v_1\}\{u_s,v_1\}]).$$

Case 2: $i=s$. Similarly,

$$R([\{u_i,v_j\}\{u_s,v_t\}])\geqslant \frac{1}{m}R^2([\{u_1,v_j\}\{u_1,v_t\}]).$$

Hence,

$$\begin{aligned}\pi(G_1\times G_2) &= \max_{[\{u_i,v_j\}\{u_s,v_t\}]} R([\{u_i,v_j\}\{u_s,v_t\}])\\ &\geqslant \max\left\{\frac{1}{n}\max_{[u_iu_s]} R^1([u_iu_s]),\ \frac{1}{m}\max_{[v_jv_t]} R^2([v_jv_t])\right\}\\ &\geqslant \max\{n\pi^{n^2}(G_1), m\pi^{m^2}(G_2)\},\end{aligned}$$

which completes the proof of the theorem. □

4. Applications

This section is an application of the above two sections to some widely used product graphs. For any natural number $n>1$, denote by G^n the product graph of n copies of graph G:

$$G^n = \underbrace{G \times G \times \cdots G}_{n}.$$

For example, K_2^n is the well known n-dimensional hypercube, which plays an important role in communication networks, where for any natural number $m > 1$, K_m is the complete graph of order n. Other known examples would be C_m^n and P_m^n, where C_m and P_m are the cycle and path of order m, respectively. These two graphs are also called the generalized cube or n-dimensional or Toroidal mesh or k-cube and mesh, etc., respectively (see [3,7,9]). By Theorems 2.1, 2.2 and 3.1.1, we have the following corollary.

Corollary 4.1. *For any graph G of order m,*

(1) $\beta(G^n) = \beta(G)/m^{n-1}$;
(2) $\pi(G^n) \leqslant 2m^{n-1}\pi(G)$, *with equality if* $\beta(G)\pi(G) = 2$;
(3) $\gamma(G^n) \leqslant \gamma(G)/m^{n-1}$;
(4) $\xi(G^n) \leqslant nm^{n-1}\xi(G) + m^{n-1}(mn - m - n) + 1$.

Proof. (1)–(3) are obvious, we prove (4) by induction on n. The assertion holds when $n = 1$, so let $n > 1$.

By Theorem 3.1.1 and the inductive hypothesis, we have

$$\begin{aligned}\xi(G^n) &= \xi(G^{n-1} \times G) \leqslant \xi(G^{n-1})m + (m^{n-1} - 1)(m-1) + \xi(G)m^{n-1} \\ &\leqslant ((n-1)m^{n-2}\xi(G) + m^{n-2}(m(n-1) - m - (n-1)) + 1)m \\ &\quad + (m^{n-1} - 1)(m-1) + m^{n-1}\xi(G) \\ &= nm^{n-1}\xi(G) + m^{n-1}(mn - m - n) + 1,\end{aligned}$$

which completes the proof of the corollary. □

It has been known [6,7,11] that

(1) $\beta(P_m) = (\lfloor m^2/4 \rfloor)^{-1}$, $\pi(P_m) = 2\lfloor m^2/4 \rfloor$, $\gamma(P_m) = (\lfloor (m-1)^2/4 \rfloor)^{-1}$ and $\xi(P_m) = 2\lfloor (m-1)^2/4 \rfloor$;
(2) $\beta(K_m) = 1, \pi(K_m) = 2, \gamma(K_m) = \xi(K_m) = 0$;
(3) $\beta(C_m) = 2(\lfloor m^2/4 \rfloor)^{-1}$, $\pi(C_m) = \lfloor m^2/4 \rfloor$, $\gamma(C_m) = 8/(m-2)^2$.

We notice that $\beta\pi = 2$ holds for P_m, K_m and C_m. So the edge expanding factors and edge forwarding indices for P_m^n, C_m^n and K_m^n are determined directly by Corollary 4.1. For the parameters γ and ξ, again by Corollary 4.1, we obtain the corresponding upper bounds for the above graphs. The parallel results can be obtained for $P_m \times P_n$ (called a mesh, see [7]), $K_m \times K_n$ and $C_m \times C_n$, also by Corollary 4.1. For explicitly, these results and some further results are listed in the following five propositions. And from the above discussion, it remains to prove (3) and (4) in Propositions 4.1 and 4.2; and (4) in Proposition 4.4.

Proposition 4.1. *Let $m, n, m \geqslant n$, be two positive integers, then* (1) $\beta(K_n \times K_m) = 1/m$,
(2) $\pi(K_n \times K_m) = 2m$,
(3) $\gamma(K_n \times K_m) = 1/\lfloor m/2\rfloor\lfloor n/2\rfloor + 1/\lceil m/2\rceil\lceil n/2\rceil$,
(4) $\xi(K_n \times K_m) = (m-1)(n-1)$.

Proof. (3) Let $V(K_n) = \{u_1, u_2, \dots, u_n\}; V(K_m) = \{v_1, v_2, \dots, v_m\}$. Let X be a proper subset of $V(G_1 \times G_2)$ and let $N(X)$ be the vertex-cut of $K_n \times K_m$ induced by X. For any $i \in \{1, 2, \dots, n\}$, denote

$$X_i = \{u\colon \{u, v_i\} \in X\} \quad \text{and} \quad X_i^+ = \{u\colon \{u, v_i\} \in X^+\}.$$

Recall that $X \cap X^+ = \phi$, we have

$$X_i \cap X_i^+ = \phi, \text{ for any } i \in \{1, 2, \dots, n\}.$$

And moreover, note that K_m and K_n are complete graphs, so if $X_i \neq \phi$, then $X_i^+ = \phi$.

Without loss of generality, assume

$$X_1^+ = X_2^+ = \cdots = X_k^+ = X_{k+1} = \cdots = X_n = \phi$$

and let

$$s = \max\{|X_1|, |X_2|, \dots, |X_k|\}; \quad t = \max\{|X_{k+1}^+|, |X_{k+2}^+|, \dots, |X_n^+|\}.$$

Since K_m is a complete graph and note that $X \cap X^+ = \phi$, then $s + t \leqslant m$. So

$$\begin{aligned}|N(X)| &= (m - |X_1|) + (m - |X_2|) + \cdots + (m - |X_k|)\\ &\quad + (m - |X_{k+1}^+|) + \cdots + (m - |X_n^+|)\\ &\geqslant kt + (m-t)(n-k).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{|N(X)|}{|X||X^+|} &= \frac{kt + (m-t)(n-k)}{\sum_{i=1}^k |X_i| \cdot \sum_{j=k+1}^n |X_j^+|} \geqslant \frac{kt + (m-t)(n-k)}{k(m-t)\cdot(n-k)t}\\ &= \frac{1}{(m-t)(n-k)} + \frac{1}{kt}.\end{aligned}$$

Consider the function in k, t: $\phi(k,t) = 1/(m-t)(n-k) + 1/kt$. Noting that $0 < t < m$ and $0 < k < n$, one can verify that $\phi(k,t)$ reaches the minimum value $1/\lfloor m/2\rfloor\lfloor n/2\rfloor + 1/\lceil m/2\rceil\lceil n/2\rceil$ when $k = \lfloor n/2\rfloor$ and $t = \lceil m/2\rceil$, or $t = \lfloor m/2\rfloor$ and $k = \lceil n/2\rceil$. That is,

$$\frac{1}{(m-t)(n-k)} + \frac{1}{kt} \geqslant \frac{1}{\lfloor m/2\rfloor\lfloor n/2\rfloor} + \frac{1}{\lceil m/2\rceil\lceil n/2\rceil}.$$

Conversely, we choose

$$X = X_1 \cup X_2 \cup \cdots \cup X_k,$$

where $X_i = \{\{u_1, v_i\}, \{u_2, v_i\}, \ldots, \{u_s, v_i\}\}$, $i = 1, 2, \ldots, k$, and where

(i) $k = n/2$, $s = m/2$ when m, n are even;
(ii) $k = n - 1/2$, $s = m + 1/2$ when m, n are odd;
(iii) $k = n - 1/2$, $s = m/2$ when m is even and n, odd;
(vi) $k = n/2$, $s = m - 1/2$ when m is odd and n, even.

Then the resulting vertex-cut $N(X)$ satisfies

$$\frac{|N(X)|}{|X||X^+|} = \frac{1}{\lfloor m/2 \rfloor \lfloor n/2 \rfloor} + \frac{1}{\lceil m/2 \rceil \lceil n/2 \rceil},$$

which completes the proof of (3). (4) is deduced directly from Proposition 5.1 in [6]. The proposition now follows. □

As the generalization of hypercube K_2^n, we have the following result for K_m^n.

Proposition 4.2. *Let m, n, be two positive integers, then*

(1) $\beta(K_m^n) = \frac{1}{m^{n-1}}$.
(2) $\pi(K_m^n) = 2m^{n-1}$.
(3) $\xi(K_m^n) = m^{n-1}((n-1)(m-1)-1)+1$
(4) $\gamma(K_m^n) \leqslant \begin{cases} \frac{8}{m^n} & \text{when } m \text{ is even}, \\ \frac{8(m^2+1)}{m^{n-2}(m^2-1)^2} & \text{when } m \text{ is odd}. \end{cases}$

Before proving Proposition 4.2, we first introduce the following Lemmas.

Lemma 4.1 (Sabidussi [10]). *Let G_1 and G_2 be two graphs. Let $u, u' \in G_1$ be two vertices of distance d_1, and $v, v' \in G_2$ be two vertices of distance d_2, then the distance between $\{u, v\}$ and $\{u', v'\}$ in $G_1 \times G_2$ is $d_1 + d_2$.*

Lemma 4.2 (Heydemann et al. [6]). *If a routing R of a graph G satisfies*:

(1) $|R(u, v)| = d(u, v)$ *for all $u, v \in V(G)$, where $d(u, v)$ is the distance between u and v*;
(2) *For all $u \in V(G)$, the loading $R(u)$ of u is a constant, then R is a vertex optimal routing.*

Proof (of Proposition 4.2). (3) Note that the fact $\xi(K_m) = 0$. So by Corollary 4.1 we have

$$\xi(K_m^n) \leqslant m^{n-1}(((n-1)(m-1)-1)+1). \tag{9}$$

Conversely, for each $k=1,2,\dots,n$, we construct a routing R_k in K_m^k by the following recursive steps:

Step 1: R_1: for any $u,v\in K_m$, define $R_1(u,v)=[uv]$ (i.e. the edge connecting u and v).

Step 2: $R_k,\ k\geqslant 2$: for any two vertices $\{u,v\}$ and $\{u',v'\}$ of $K_m\times K_m^{k-1}=K_m^k$, define

$$R_k(\{u,v\},\{u',v'\})=\{u,v\}\{u,v_1\}\cdots\{u,v_q\}\{u,v'\}\{u',v'\},$$

where the path $vv_1\cdots v_qv'=R_{k-1}(v,v')$. Then it can be seen easily that for any vertex $w\in K_m^n$, the number of paths of R_n admitting w as an inner point is a constant, i.e.

$$R_n(w)=m^{n-1}((n-1)(m-1)-1)+1.$$

By Lemma 4.1, it is not difficult to verify that for any $w,w'\in G_m^n$, the length of $R_n(w,w')$ equals the distance between w and w'. So by Lemma 4.2, R_n is a vertex optimal routing. Combine with (9), we complete the proof of (3).

(4) Let us apply induction on n. By Proposition 4.1. (3) the assertion holds when $n=2$. By Theorem 2.2 and the inductive hypothesis,

$$\gamma(K_m^n)=\gamma(K_m^2\times K_m^{n-2})\leqslant\frac{\gamma(K_m^2)}{|V(K_m^{n-2})|}\leqslant\begin{cases}\frac{8}{m^n} & \text{when } m \text{ is even},\\ \frac{8(m^2+1)}{m^{n-2}(m^2-1)^2} & \text{when } m \text{ is odd}.\end{cases}$$

The proof of Proposition 4.2 is now completed. □

So far the value of $\gamma(K_m^n)$ still remains open, even for $m=2$, i.e. a hypercube. Let u_0 be a fixed vertex in K_2^n, one can find a better upper bound (than that in Proposition 4.2) of $\gamma(K_2^n)$ by choosing the vertex-cut ∂X induced by

$$X=\begin{cases}\{u\colon d(u,u_0)<n/2\} & \text{when } n \text{ is even},\\ \{u\colon d(u,u_0)<(n-1)/2\} & \text{when } n \text{ is odd}\end{cases}$$

which yields

$$\gamma(K_2^n)\leqslant\begin{cases}\frac{4C_n^{n/2}}{(2^n-C_n^{n/2})^2} & \text{when } n \text{ is even},\\ \frac{C_n^{(n-1)/2}}{2^{n-1}(2^{n-1}-C_n^{(n-1)/2})} & \text{when } n \text{ is odd},\end{cases}\tag{10}$$

where for two vetices u and v, $d(u,v)$ denotes the distance between u and v.

Conjecture 4.1. *The* '$\leqslant$' *in* (10) *is* '$=$'.

Chung et al. [3] determined the vertex forwarding index of C_m^n (in their paper, C_m^n is called m-cube and is denoted by K_{2n,m^n}). For other parameters of C_m^n, we have

Proposition 4.3. (1) $\beta(C_m^n)=2/m^{n-1}\lfloor m^2/4\rfloor$ *and* $\pi(C_m^n)=m^{n-1}\lfloor m^2/4\rfloor$.

(2) $\gamma(C_m^n)\leqslant 8/(m-2)^2m^{n-1}$.

Proposition 4.4. (1) $\beta(C_m \times C_n) = \min\{2/m\lfloor n/2\rfloor\lceil n/2\rceil, 2/n\lfloor m/2\rfloor\lceil m/2\rceil\}$.
(2) $\pi(C_m \times C_n) = \max\{m\lfloor n^2/4\rfloor, n\lfloor m^2/4\rfloor\}$.
(3) $\gamma(C_m \times C_n) \leqslant \min\{2/m\lfloor n/2\rfloor\lceil n/2\rceil, 2/n\lfloor m/2\rfloor\lceil m/2\rceil\}$.
(4) $\xi(C_m \times C_n) = m\lfloor (n-2)^2/4\rfloor + (m-1)(n-1) + n\lfloor (m-2)^2/4\rfloor$.

Proof. (4) Let R be a routing in $C_m \times C_n$ satisfying for any two vertices $\{u,v\},\{u',v'\} \in V(C_m \times C_n)$,

$$R(\{u,v\},\{u',v'\}) = \{u,v\}\{u,v_1\}\cdots\{u,v_q\}\{u,v'\}\{u_1,v'\}\cdots\{u_p,v'\}\{u',v'\},$$

where $uu_1\cdots u_pu'$ is the shortest path connecting u and u' (ordered pair) in C_m (if the distance between u and u' is $m/2$, then the path connecting u and u' is defined to be one of the shortest two paths while the path connecting u' and u is the other) and $vv_1\cdots v_qv'$ is the shortest path connecting v and v' in C_n. By the same discussion as in the proof of Proposition 4.2, (3), one can check easily that R is a vertex optimal routing with $R(w) = m\lfloor (n-2)^2/4\rfloor + (m-1)(n-1) + n\lfloor (m-2)^2/4\rfloor$ for all $w \in V(C_m \times C_n)$. So by Theorem 3.1.1, we complete the proof. □

Proposition 4.5. (1) $\beta(P_m^n) = (m^{n-1}\lfloor m^2/4\rfloor)^{-1}$ *and* $\pi(P_m^n) = 2m^{n-1}\lfloor m^2/4\rfloor$.
(2) $\gamma(P_m^n) \leqslant (m^{n-1}\lfloor (m-1)^2/4\rfloor)^{-1}$ *and* $\xi(P_m^n) \leqslant 2nm^{n-1}\lfloor (m-1)^2/4\rfloor + m^{n-1}(mn - m - n) + 1$.

5. Final remark

Contrast to the edge expanding factor of product graphs, we only obtain some lower and upper bounds for other parameters. In fact, there actually exist some product graphs which meet the inequalities in Theorems 2.2 or 3.1.1. For example, let Q be the graph: $V(Q)=\{q_{i,j}: i=1,2; j=1,2,3,4\}$ and $E(Q)=\{[q_{i,j}q_{s,t}]: i=s \text{ or } j=t \in \{1,2,3\}\}$. Then one can check that $\pi(Q)=11$ and $\pi(Q\times K_3)=32 < \max\{\pi(Q)|V(K_3)|, \pi(K_3)|V(Q)|\}=33$. Based on the fact that $\pi(G) \geqslant \pi^t(G)$ and $\xi(G) \geqslant \xi^t(G)$, it is possible to decrease the load of vertices or edges by dividing the message into t blocks. For example, we have $\pi(Q) > \pi^3(G) = \frac{32}{3}$. But given a natural number t, which graphs satisfy the above inequality (or equality) is still unknown. We give the following two open problems to end the paper.

Open problem 1. *Which graphs satisfy* $\xi(G) = \xi^0(G)$ *or* $\pi(G) = \pi^0(G)$?

Open problem 2. *Which product graphs meet the equality in Theorems* 2.2 *or* 3.1.1?

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